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One-dimensional Ising models with long-range interactions

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Abstract. We consider Ising models with long-range ferromagnetic pair interactions decaying as $1/r^{\theta}$ for $1.0 < \theta \leqslant 1.5$. We first find approximate values for the critical temperature. We use a cluster mean-field approach combined with finite-size scaling and Vanden Broeck and Schwartz transformations. For $\theta = 1.10$ we find $T_c = 21.000\,97$ which can be compared with recent results of Luijten and Blöte who found $T_c = 21.000\,99 \pm 0.000\,26$, and which is two orders of magnitude more accurate than any previous results. Since we use a mean-field cluster approximation as part of our approach, the accuracy for larger values of θ decreases significantly. In addition to T_c we obtain approximate values for the critical exponents β, γ and δ using the coherent anomaly method. For $\theta = 1.10$ we obtain $\beta = 0.4995$, $\gamma = 1.0008$, and $\delta = 2.9947$—all extremely close to the predictions of renormalization group calculations which say that these exponents should take on their classical values for this value of θ.

1. Introduction

In 1969 Dyson [1] proved the existence of a phase transition for a one-dimensional Ising model with long-range ferromagnetic pair interactions decaying as $1/r^{\theta}$ with $1 < \theta < 2$. Not long after, specifically 1970, Nagle and Bonner [2] made the first numerical approximations of the critical temperature, T_c, and critical exponents for these models. Since then a stream of rigorous results and numerical estimates of the critical temperatures and critical exponents have appeared. An excellent review of these results has recently appeared in a paper by Luijten and Blöte [3]. In addition to the review of past results these two authors have performed extensive Monte Carlo simulations of these systems resulting in estimates of both the critical temperature and critical exponents. These results are limited to the case where $1 < \theta \leqslant 1.50$. They point out that their critical temperature estimates are two orders of magnitude more accurate than previous estimates. This large increase in the accuracy of T_c estimates has caused the present author to re-examine and extend some previous work by himself, Lucente and Hourlland [4]. This work involved the use of the coherent anomaly method (CAM) of Suzuki [5] and cluster mean-field estimates to obtain approximate values for the critical temperature and the critical exponents β and γ. Here we retain the cluster mean-field approach but combine it with a finite-size scaling approach in combination with methods to accelerate the convergence of finite-lattice sequences, rather than the CAM, to increase the accuracy of our critical temperature estimates by several orders of magnitude. We restrict ourselves to the case, as done by Luijten and Blöte, where $1 < \theta \leqslant 1.50$. For very slowly varying interactions, e.g. $\theta = 1.10$, we obtain accuracy at least equal to that of Luijten and Blöte. After estimating T_c we go back to the CAM to obtain estimates for

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the critical exponents β, γ and δ. With this approach we also increase the accuracy of our critical exponent estimates. This increased accuracy is substantial when the interaction falls off very slowly but rather minor when this is not the case.

In the following section we present the necessary notation as well as the approach used to generate the 'data' used to obtain our final critical temperature and critical exponent estimates. This is followed by section 3 with our critical temperature results and by section 4 with our results for the critical exponents β, γ and δ.

2. Notation and mean-field estimates

We consider a one-dimensional lattice of sites with Hamiltonian

$$H(\{\sigma\}) = -\sum_{i<j} \frac{J}{|i-j|^{\theta}} \sigma_i \sigma_j - h \sum_i \sigma_i \tag{1}$$

where σ_i is the spin variable on the ith site, $\sigma = \pm 1$, $\{\sigma\}$ denotes a configuration of the system, and $|i-j|$ is the distance between sites i and j with the distance between adjacent sites set equal to one. Hereafter, J, the interaction strength, will be set equal to one. J positive means we have a ferromagnetic system. The thermal average of a spin is defined as

$$\langle \sigma_i \rangle = Z^{-1} \sum_{\{\sigma_i\}} \sigma_i \exp[-\beta H(\{\sigma_i\})] \tag{2}$$

where Z is the partition function, the sum is over all configurations, and $\beta = 1/kT$. Hereafter we set k, the Boltzmann constant, equal to one.

Our methods of section 3 require a sequence of critical temperature estimates for the above system (of course for the determination of the critical temperature we take $h = 0$) and we achieve this by use of the cluster mean-field approach. Here we treat exactly all interactions among the spins making up a cluster and we replace all interactions between a spin in the cluster and one outside the cluster with a mean-field interaction. As an example we have for a three-site cluster

$$\begin{aligned} H(\sigma_1, \sigma_2, \sigma_3) = {} & -J[\sigma_1\sigma_2 + \sigma_2\sigma_3] - \frac{J}{2^{\theta}}\sigma_1\sigma_3 \\ & -Jm(\sigma_1 + \sigma_3)\left[\sum_{n=1}^{\infty} \frac{1}{n^{\theta}} + \sum_{n=3}^{\infty} \frac{1}{n^{\theta}}\right] - Jm\sigma_2\left[2\sum_{n=2}^{\infty} \frac{1}{n^{\theta}}\right] \end{aligned} \tag{3}$$

where m represents the mean field. We then require that the thermal average of the spin in the middle of our cluster equal m, i.e. for the above case $\langle \sigma_2 \rangle = m$. For temperatures greater than the mean-field critical temperature the only solution occurring is $m = 0$. However, as the temperature is lowered there occur solutions with $m \neq 0$. The temperature below which non-zero solutions exist is the mean-field critical temperature for that cluster size. We denote this critical temperature as $T_c(L)$, the L representing the number of sites. We look at clusters with odd numbers of sites from 1 to 25. In table 1 we list the values of $T_c(L)$ for clusters of 1 to 25 sites for $\theta = 1.1$ and $\theta = 1.5$. One notices that $T_c(L)$ decreases monotonically with cluster size. It is also worth mentioning that these $T_c(L)$ values are rigorous upper bounds on the critical temperature of the infinite system [6, 7]. We give $T_c(L)$ values to 16 figures past the decimal because, as we shall see in section 3, one needs $T_c(L)$ values to 16 or more figures if one does not want to limit the obtainable accuracy found by the methods to be presented.

Table 1. $T_c(L)$ values for clusters of $1, 3, \ldots, 23$ and 25 sites for $\theta = 1.10$ and 1.50.

L	$\theta = 1.1$	$\theta = 1.5$
1	21.168 896 929 901 6197	5.224 750 697 370 9767
3	21.078 195 053 616 4672	4.893 079 043 100 1155
5	21.051 934 167 793 0418	4.769 602 703 082 4686
7	21.039 390 542 274 3332	4.701 743 199 107 3253
9	21.031 994 058 422 2680	4.657 709 855 152 5970
11	21.027 092 374 695 5513	4.626 343 454 814 8782
13	21.023 593 263 559 3630	4.602 619 275 207 2211
15	21.020 963 291 404 3735	4.583 908 214 597 7935
17	21.018 910 351 341 6567	4.568 687 747 585 7609
19	21.017 260 744 701 0522	4.556 008 948 358 6706
21	21.015 904 540 759 9219	4.545 246 449 189 2952
23	21.014 768 683 725 5628	4.535 969 626 602 2390
25	21.013 802 675 396 8671	4.527 871 257 058 7719

For the estimation of critical exponents we use the CAM of Suzuki [5]. Since our results are mean-field results we know if we look at the spontaneous magnetization, m_s, we have

$$m_s(L) = \bar{m}_s(L)|\varepsilon|^{1/2} \qquad \varepsilon \equiv \frac{T - T_c(L)}{T_c(L)} \tag{4}$$

where ε is to the power $\frac{1}{2}$ which is the classical value for the critical exponent β. In a similar fashion for the zero-field susceptibility, $\chi(L)$, one has

$$\chi(L) = \bar{\chi}(L)\frac{1}{\varepsilon} \tag{5}$$

and for the magnetization at the critical temperature as a function of the magnetic field h, $m_c(L)$, one has

$$m_c(L) = \bar{m}_c(L)h^{1/3}. \tag{6}$$

Suzuki's CAM method makes use of $\bar{m}_s(L)$, $\bar{\chi}(L)$ and $\bar{m}_c(L)$ to determine the true, and thus not necessarily classical, critical exponent values of β, γ and δ. The values are given by

$$\beta = \frac{1}{2} - \frac{\log(\bar{m}_s(L_1)/\bar{m}_s(L_2))}{\log(\rho)} \tag{7}$$

$$\gamma = 1 + \frac{\log(\bar{\chi}(L_1)/\bar{\chi}(L_2))}{\log(\rho)} \tag{8}$$

$$\frac{\gamma(\delta - 3)}{3(\delta - 1)} = \frac{\log(\bar{m}_c(L_1)/\bar{m}_c(L_2))}{\log(\rho)} \tag{9}$$

where L_1 and L_2 denote two different cluster sizes and where

$$\rho = \frac{T_c(L_2) - T_c}{T_c(L_1) - T_c} \tag{10}$$

with T_c the true critical temperature for the system being investigated. Knowing $T_c(L)$ and either $\bar{m}_s(L)$, $\bar{\chi}(L)$, or $\bar{m}_c(L)$ for three different cluster sizes then T_c and one of the critical exponent values can be determined. This we did in our earlier paper [4]. We now use a finite-size approach to first get an approximation for the true critical temperature and then we use equations (7)–(9) to obtain values for β, γ and δ. This greatly increases the accuracy of our results.

Table 2. T_c estimates using equation (11) and the three clusters listed in the left column.

Cluster sequence used	$\theta = 1.1$	$\theta = 1.5$
1,3 & 5 sites	20.974 656 76	3.943 302 24
3,5 & 7 sites	20.995 717 12	4.260 157 14
5,7 & 9 sites	20.998 771 59	4.312 017 66
7,9 & 11 sites	20.999 756 18	4.331 703 07
9,11 & 13 sites	21.000 190 84	4.341 622 28
11,13 & 15 sites	21.000 420 25	4.347 434 75
13,15 & 17 sites	21.000 556 31	4.351 180 55
15,17 & 19 sites	21.000 643 93	4.353 758 49
17,19 & 21 sites	21.000 703 92	4.355 620 33
19,21 & 23 sites	21.000 746 96	4.357 015 55
21,23 & 25 sites	21.000 779 04	4.358 092 05

3. Critical temperature estimates

As the notation $T_c(L)$ indicates, the mean-field critical temperature is dependent on the size of the cluster. Using a finite size scaling [8] approach the convergence of the mean-field critical temperatures to the true critical temperature can be written as

$$\frac{T_c(L) - T_c}{T_c} \approx \frac{b}{L^\lambda} \tag{11}$$

where λ is the shift exponent. Hence knowing $T_c(L)$ for three different cluster sizes allows one to compute an approximation to T_c. We have in table 2 results for $\theta = 1.1$ and $\theta = 1.5$ using three cluster sequences of 1, 3 and 5 sites to 21, 23 and 25 sites. We list our estimates of $T_c(L)$ to eight places past the decimal only to better illustrate the systematic increase in the T_c values and not to imply that this is the accuracy of the results.

Regarding accuracy, we note that Luijten's and Blöte's critical temperature value for $\theta = 1.1$ is $T_c = 21.000\,99 \pm 0.000\,26$ and for $\theta = 1.5$ they have $T_c = 4.3638 \pm 0.0001$. Thus we see that for the very slowly decaying case of $\theta = 1.1$ our approximation is already within Luijten's and Blöte's error bounds but for $\theta = 1.5$ our value is significantly below their error bounds. In [4] where the cluster mean-field results along with the CAM were used for a cluster sequence consisting of 13, 15 and 17 sites (the largest examined in that reference) we found that for $\theta = 1.1$ the T_c estimate was 20.959 when using equation (7) and 20.908 when using equation (8), while for $\theta = 1.5$ the T_c estimates were $T_c = 4.363$ and $T_c = 4.283$ using equations (7) and (8), respectively. For $\theta = 1.1$ the present results are clearly better while for $\theta = 1.5$ the 4.363 found using equation (7) coincides closely with the Luijten and Blöte result, while the approach using equation (8) is quite far off. We thus suspect that the accuracy obtained using equation (7) for the $\theta = 1.5$ case is misleading.

What is particularly evident from table 1 is the monotonic decrease in the value of T_c with the increase in the cluster sizes used. We repeat that these are known to be upper bounds on T_c [6, 7] and improve as the size of the mean-field cluster increases. What is particularly evident in table 2 is the monotonic increase in the T_c values and we conjecture that the results are lower bound for T_c. In table 5 we present results using the 21, 23 and 25 site-cluster sequence for various θ values in the interval $1.1 < \theta \leqslant 1.5$. One sees that for all θ the T_c values given by equation (11) and the data from the 21, 23 and 25 site clusters is below that given by Luijten and Blöte and as θ increases the difference between the two values increases.

One can further improve the above results if one uses the sequence transformation methods introduced into statistical mechanics by Hamer and Barber [9]. These are techniques used to accelerate the convergence of sequences of the type given by the $T_c(L)$'s. The sequence transformations are originally due to Vanden Broeck and Schwartz [10]. Using the notation of Hamer and Barber [9] one has for the general sequence transformation that given a sequence of values A_L which converge to a limiting value A_∞ one forms a table of approximants to A_∞ denoted by $[L, N]$ where $[L, 0] = A_L$ and the $(N + 1)$th column of approximants is generated from the Nth and $(N - 1)$th columns via the formula

$$\frac{1}{[L, N+1] - [L, N]} + \frac{\alpha_N}{[L, N-1] - [L, N]} = \frac{1}{[L+1, N] - [L, N]} + \frac{1}{[L-1, N] - [L, N]} \tag{12}$$

with $[L, -1] \equiv \infty$. Again following Hamer and Barber we refer to these approximants as VBS approximants.

The above defines a broad class of transformations based on the definition of α_N. For the case where the sequence converges as

$$A_L \approx A_\infty + b_1 L^{-\lambda_1} + b_2 L^{-\lambda_2} + \cdots. \tag{13}$$

Barber and Hamer [11] show that a good choice for the value of α_N is

$$\alpha_N = -\frac{[1 - (-1)^N]}{2} \tag{14}$$

for $N = 0, 1, 2, \ldots$. The table of approximants using this approach are given in tables 3 and 4 for $\theta = 1.1$ and $\theta = 1.5$, respectively. The more terms in the original sequence of $T_c(L)$ available, the more accuracy is needed for these terms. For our sequence consisting of 13 terms (see table 1) we need the $T_c(L)$ values determined to a minimum of 16-figure accuracy. A single change in the 16th figure of one of the terms of the sequence influences our final estimate of T_c in the seventh figure which is what we believe to be close to our accuracy for the $\theta = 1.1$ case. For higher θ values we have less accuracy. The entries in tables 3 and 4 are based on 18-figure accuracy for all $T_c(L)$'s.

Using the VBS transformations for $\theta = 1.1$ we obtain as our estimates for $T_c = 21.000\,97$ which is almost identical to the Luijten and Blöte result of $21.000\,99 \pm 0.000\,26$. As is not surprising, since we are using a mean-field approach to get our initial input, for

Table 3. Table of VBS approximants of T_c for $\theta = 1.10$ using α_N defined in equation (14). All calculations were done to 18-figure accuracy though only the first 12 digits are given in the table. For the full 18 figures for the left-hand column see table 1.

21.168 896 9299						
21.078 195 0536	21.041 232 3750					
21.051 934 1678	21.027 920 1365	20.999 984 6914				
21.039 390 5423	21.021 365 2518	21.000 594 2846	21.000 847 0657			
21.031 994 0584	21.017 461 7421	21.000 772 9699	21.000 900 1543	21.000 956 4674		
21.027 092 3747	21.014 863 7484	21.000 847 2695	21.000 922 6479	21.000 958 7574	21.000 962 5818	
21.023 593 2636	21.013 005 1240	21.000 884 6870	21.000 934 3552	21.000 960 1898	21.000 963 5117	21.000 965 7619
21.020 963 2914	21.011 606 4891	21.000 906 0276	21.000 941 2873	21.000 961 1906	21.000 964 0609	
21.018 910 3513	21.010 513 9642	21.000 919 3220	21.000 945 7743	21.000 961 9327		
21.017 260 7447	21.009 635 7193	21.000 928 1696	21.000 948 8746			
21.015 904 5408	21.008 913 5019	21.000 934 3684				
21.014 768 6837	21.008 308 5382					
21.013 802 6754						

Table 4. Table of VBS approximants of T_c for $\theta = 1.40$ using α_N defined in equation (14). All calculations were done to 18-figure accuracy though only the first 12 digits are given in the table. For the full 18 figures for the left-hand column see table 1.

5.224 750 697 37						
4.893 079 043 10	4.696 371 436 57					
4.769 602 703 08	4.618 946 107 30	4.363 375 411 91				
4.701 743 199 11	4.576 331 430 00	4.364 247 903 90	4.364 681 691 25			
4.657 709 855 15	4.548 672 700 14	4.364 537 639 78	4.364 759 586 09	4.364 766 598 72		
4.626 343 454 81	4.528 970 957 80	4.364 663 315 07	4.364 765 838 31	4.364 762 231 60	4.364 767 025 68	
4.602 619 275 21	4.514 070 696 43	4.364 719 777 48	4.364 756 541 51	4.364 713 195 93	4.364 732 273 25	4.364 750 076 01
4.583 908 214 60	4.502 320 062 66	4.364 742 043 54	4.364 746 873 50	4.364 744 421 59	4.364 744 802 32	
4.568 687 747 59	4.492 762 307 40	4.364 746 012 54	4.364 743 586 60	4.364 744 797 74		
4.556 008 948 36	4.484 801 119 71	4.364 739 772 62	4.364 752 742 23			
4.545 246 449 19	4.478 043 535 33	4.364 727 746 89				
4.535 969 626 60	4.472 218 985 11					
4.527 871 257 06						

Table 5. The T_c estimates based on equation (11) and using clusters of 21, 23 and 25 sites, based on the VBS transformations, and the results of [3].

θ	Using equation (11) and a three cluster sequence	Using the alternating alpha VBS transformation	Results from Luijten and Blöte [3]
1.01	201.139 389	201.139 389	
1.04	51.093 79	51.093 85	
1.07	29.618 9	29.619 12	
1.10	21.007	21.000 97	21.000 99 ± 0.000 26
1.20	10.841 1	10.842 0	10.842 29 ± 0.000 2
1.30	7.344 9	7.347 2	7.347 0 ± 0.000 1
1.40	5.516	5.520 2	5.520 3 ± 0.000 1
1.50	4.358	4.364 7	4.363 8 ± 0.000 1

increasing θ our estimates become less accurate. For $\theta = 1.5$ we obtain as our estimate $T_c = 4.365$ while Luijten and Blöte obtain 4.3638 ± 0.0001. Results for $\theta = 1.1$, 1.2, 1.3, 1.4 and 1.5 are given in table 5 along with the results of Luijten and Blöte for these five cases.

As Hamer and Barber point out, the apparent convergence of the VBS tables can sometimes be misleading, especially with respect to the accuracy of the estimates. In their original work they were able to 'M-shift' their sequences which allowed them to obtain some idea of the accuracy of their results. Unfortunately we have been unable to implement this scheme for our T_c estimates.

Based on the fact that our method becomes increasingly accurate as $\theta \to 1$, it is natural that we should consider the conjecture of Cannas [12] that one has

$$\lim(\theta \to 1)\frac{1}{T_c} \approx \frac{\theta - 1}{2}. \qquad (15)$$

We have looked at the following θ values, 1.07, 1.04 and 1.01, and the results for these θ values are presented in table 5. For these results we have computed $T_c(L)$ only for cluster sizes up to and including 17 sites and not the 25 sites done for the other θ values. Nevertheless we see that the estimate for $\theta = 1.01$ is accurate to approximately eight figures even for this abbreviated sequence of clusters. We also see that our results support the conjecture of Cannas.

4. Critical exponents

In this section we obtain estimates of the critical exponents β, γ and δ, using equations (7)–(9) along with the VBS transformations. We find that these methods require us to know $\bar{m}_s(L)$, $\bar{\chi}(L)$, and $\bar{m}_c(L)$ to approximately 12-figure accuracy. This is particularly true for small θ values where our results for the critical exponents have three- and four-figure accuracy. Since we obtain these quantities by calculating the spontaneous magnetization, the zero-field susceptibility, and the magnetization as a function of h at the critical temperature and then using equations (4)–(6), respectively, we need to know the critical temperature, $T_c(L)$, to extreme accuracy. For all the following results we used $T_c(L)$ values accurate to 30 figures. This level of accuracy would not be needed if we did not employ the VBS transformations but these transformations significantly improve our estimates of the critical exponents as they did with the critical temperature estimates of the previous section. Because we need this level of accuracy we have, for the critical exponents, used clusters whose maximum size is 15 sites. Nevertheless we will see that, especially for small θ, we obtain accurate estimates of the critical exponents.

The general procedure is to use equations (4)–(6) to get $\bar{m}_s(L)$, $\bar{\chi}(L)$, and $\bar{m}_c(L)$ for clusters whose number of sites are $1, 3, 5, \ldots, 15$. Then, using pairs of clusters consisting of 1 and 3 site clusters, 3 and 5 site clusters, up to a pair consisting of 13 and 15 site clusters and the coherent anomaly equations (7)–(9), we obtain a sequence of seven estimates for each critical exponent. These estimates are all listed in table 6. For T_c needed in equations (7)–(9) we use the T_c found in the previous section. Then using these sequences of seven estimates we use the VBS transformation with $\alpha_N = 0$ for all N and do not use the alternating value of α_N used in the previous section to obtain our final best estimate of the critical exponent values.

As with our estimates of T_c, the smaller is θ, the more accurate our estimates. For $\theta = 1.10$ we find $\beta \equiv 0.4995$, $\gamma \cong 1.0008$, and $\delta \cong 2.9947$. This is to be compared with the results of [4], in which for $\theta = 1.10$ it was reported that $\beta \cong 0.495$ and $\gamma \cong 1.014$, and no estimate for δ was given. Using the values of $y_t = 0.507$ and $y_h = 0.7493$ of [3] one obtains $\beta \cong 0.4945$, $\gamma \cong 0.9843$, and $\delta \cong 2.9888$. Our values are seen to be more accurate for this value of θ. However, as θ increases we quickly lose accuracy and for $\theta = 1.50$ we have $\beta \cong 0.408$, $\gamma \cong 1.13$, and $\delta \cong 2.488$. The results of [3] for this θ value are $y_t = 0.501$ and $y_h = 0.7492$ giving $\beta \cong 0.5006$, $\gamma \cong 0.9948$, and $\delta \cong 2.987$. It should be pointed out that the results of [3] include error bars on y_t and y_h and these error bars, in general, do increase as θ increases but not to the extent that inaccuracies increase in the method of this paper. Final results for the three critical exponents considered here are given in table 7. In the case of δ, equation (9), the coherent anomaly equation we have used to estimate δ, also involves the exponent γ. We have used our estimates for γ found using equation (8) in equation (9) to determine δ and we did not assume $\gamma = 1$ and then calculate δ. Hence our method is completely self-contained and we make no assumptions about one critical exponent in order to calculate another.

A couple of cautionary remarks are warranted. First, for $\theta = 1.10$ one can see from table 6 that for all three critical exponents the sequence of seven values given by the coherent anomaly method are monotonically increasing in the cases involving β and δ, and decreasing in the case of γ. In all cases moving toward the classical values predicted by renormalization group methods. However, when θ increases this is not always the case. For example, for $\theta = 1.40$ and the β exponent, the value given by the estimate using clusters of 3 and 5 sites is farther from the classical value of $\frac{1}{2}$ than that obtained using 1 and 3 site clusters. After this, as one looks at larger cluster pairs, the estimates all increase and

Table 6. Critical exponent approximations using CAM and T_c values from the previous table. A contains the results for β, B contains the results for γ, and C contains the results for δ.

A

Number of sites in cluster pairs ↓	$\theta = 1.10$	$\theta = 1.20$	$\theta = 1.30$	$\theta = 1.40$	$\theta = 1.50$
1 & 3	0.495 146	0.482 301	0.463 650	0.444 277	0.416 340
3 & 5	0.496 682	0.485 381	0.466 319	0.435 487	0.412 600
5 & 7	0.497 365	0.487 078	0.468 094	0.441 771	0.410 653
7 & 9	0.497 774	0.488 247	0.469 504	0.442 428	0.409 628
9 & 11	0.498 053	0.489 127	0.470 675	0.443 135	0.409 082
11 & 13	0.498 259	0.489 824	0.471 673	0.444 503	0.408 805
13 & 15	0.498 417	0.490 397	0.472 538	0.445 137	0.408 692

B

Number of sites in cluster pairs ↓	$\theta = 1.10$	$\theta = 1.20$	$\theta = 1.30$	$\theta = 1.40$	$\theta = 1.50$
1 & 3	1.010 837	1.043 277	1.096 077	1.166 711	1.251 761
3 & 5	1.006 594	1.030 779	1.076 326	1.143 820	1.231 399
5 & 7	1.004 939	1.025 149	1.066 475	1.131 464	1.219 607
7 & 9	1.004 019	1.021 741	1.060 121	1.123 070	1.211 175
9 & 11	1.003 423	1.019 391	1.055 537	1.116 788	1.204 632
11 & 13	1.003 001	1.017 645	1.052 010	1.111 816	1.199 311
13 & 15	1.002 683	1.016 282	1.049 176	1.107 732	1.194 845

C

Number of sites in cluster pairs ↓	$\theta = 1.10$	$\theta = 1.20$	$\theta = 1.30$	$\theta = 1.40$	$\theta = 1.50$
1 & 3	2.959 770	2.854 937	2.716 884	2.578 184	2.466 150
3 & 5	2.973 903	2.887 393	2.753 788	2.605 870	2.473 693
5 & 7	2.979 793	2.903 511	2.774 157	2.622 175	2.481 271
7 & 9	2.983 214	2.913 917	2.788 377	2.634 280	2.487 745
9 & 11	2.985 485	2.921 429	2.799 265	2.643 998	2.492 773
11 & 13	2.987 127	2.927 210	2.808 418	2.652 129	2.497 460
13 & 15	2.988 380	2.931 851	2.815 360	2.659 118	2.501 649

Table 7. Critical exponent values found using VBS transformations with $\alpha_N = 0$.

	$\theta = 1.10$	$\theta = 1.20$	$\theta = 1.30$	$\theta = 1.40$	$\theta = 1.50$
β	0.499 507	0.496 10	0.464 58	0.439 94	0.408 43
γ	1.000 8	1.006 0	1.023	1.064	1.137
δ	2.994 7	2.973 4	2.907	2.896	2.535

move toward the classic value. In the case of $\theta = 1.50$, for our entire seven-term sequence, the values of β decrease, moving away from the classic value. Their decrease is slower as the cluster sizes increase, so we believe that they may eventually begin to increase as in the $\theta = 1.40$ case. The second remark is that the very systematic properties shown for our VBS transformations of T_c as shown in tables 3 and 4 are continued in the calculations for the critical exponents only for the case $\theta = 1.10$ and become less systematic as θ increases.

5. Conclusion

In the above we have shown that when the decay rate for ferromagnetic interactions in a long-range one-dimensional Ising model is small, one can obtain very accurate estimates for the critical temperature using finite-size scaling and VBS transformations. Similarly, one can obtain very accurate estimates of the critical exponents using the coherent anomaly method and VBS transformations. In particular, for $\theta = 1.10$, the accuracy of the results equals or surpasses the most accurate results known to date: those of Luijten and Blöte given in [3]. As θ increases, the accuracy decreases significantly.

As with any approximation, its value is dependent to some extent on the work required by the approximation method. All of the above computations were performed on a personal computer using Mathematica and they required about one month of computer time to produce the above results. Obviously much larger clusters could be considered resulting in improved accuracy if larger computer resources were to be used.

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